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A Class of Even and Odd Nonlinear Coherent States and Their Properties

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A class of even and odd nonlinear coherent states are introduced. The properties of some related states, including quadrature squeezing, antibunching effect and phase probability distribution, are studied.

KEY WORDS: even and odd nonlinear coherent states; quadrature squeezing; antibunching effect; Pegg–Barnett formula; phase probability distribution.

1. INTRODUCTION

Coherent states (CSs) are important in many fields of physics (Klauder and Skagerstam, 1985; Zhang *et al.*, 1990). CSs $|\alpha\rangle$, defined as the eigenstates of the harmonic oscillator annihilation operator $\hat{a}$, $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$ (Glauber, 1963), have properties like the classical radiation field. There exist states of the electromagnetic field whose properties, like squeezing, higher-order squeezing, antibunching and sub-Poissonian statistics (Walls, 1983; Loudon and Knight, 1987), are strictly quantum mechanical in nature. These states are called nonclassical states. The CSs define the limit between the classical and nonclassical behavior of the radiation field as far as the nonclassical effects are considered.

Recently there has been much interest in the study of nonlinear coherent states (NLCSs) called f-CSs (de Matos Filho and Vogel, 1996; Man'ko *et al.*, 1997), which are eigenstates of the annihilation operator $f(\hat{n})\hat{a}$ of f-oscillators, where $f(\hat{n})$ is an operator-valued function of the boson number operator $\hat{n} = \hat{a}^{+}\hat{a}$. A class of f-CSs can be realized physically as the stationary states of the center-of-mass motion of a trapped ion (de Matos Filho and Vogel, 1996). The NLCSs exhibit nonclassical features such as squeezing and self-splitting. Subsequently, a new kind of NLCSs was constructed by Roy and Roy (2000). They are defined as

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the eigenstates $|\beta, f\rangle$ of the operator $\hat{B} = \frac{1}{f(\hat{n})}\hat{a}$. In the number-state basis, $|\beta, f\rangle$ is given by (Roy and Roy, 2000; Choquette *et al.*, 2003)

$$|\beta, f\rangle = N_f \sum_{n=0}^{\infty} \frac{\beta^n f(n)!}{\sqrt{n!}}|n\rangle, \quad N_f = \left\{\sum_{n=0}^{\infty} \frac{|\beta|^{2n}[f(n)!]^2}{n!}\right\}^{-1/2}, \tag{1}$$

where β is an arbitrary complex number. $f(n)! = f(n)f(n-1)\cdots f(1)f(0)$ and $f(0) = 1$.

The eigenstates of the operator $\hat{a}^2$ show some of the nonclassical features. These eigenstates can be written as a combination of the CSs $|\alpha\rangle$ and $|-\alpha\rangle$ (Dodonov *et al.*, 1974). The symmetric combination is the even coherent state (ECS), $|\alpha\rangle_+$, and its number-state expansion is

$$|\alpha\rangle_+ = (\cosh|\alpha|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{\sqrt{(2n)!}}|2n\rangle. \tag{2}$$

The antisymmetric combination is the odd coherent state (OCS), $|\alpha\rangle_-$, given by

$$|\alpha\rangle_- = (\sinh|\alpha|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{\sqrt{(2n+1)!}}|2n+1\rangle. \tag{3}$$

The ECS has a squeezing but no antibunching effect. The OCS has an antibunching effect but no squeezing effect (Hillery, 1987; Xia and Guo, 1989). The notion of even and odd CSs has been generalized to the case of NLCSs as the superposition of the f-CSs $|\alpha, f\rangle$ and $|-\alpha, f\rangle$ (Mancini, 1997). These states are the eigenstates of the operator $\hat{A}^2$ where $\hat{A}$ is the generalized annihilation operator $f(\hat{n})\hat{a}$. In the present paper, we introduce a different class of even and odd NLCSs. The sum of their properties, including quadrature squeezing, antibunching effect and phase probability distribution, are studied. Because of these effects they have the typical nonclassical properties. It is shown that the nonclassical properties of the class of even and odd NLCSs are very different from those of the usual even and odd CSs.

2. DEFINITION OF EVEN AND ODD NLCSs

According to the definition of the even and odd NLCSs (Mancini, 1997; Sivakumar, 1998), a class of new even and odd NLCSs are defined as eigenstates of the operator $\frac{1}{F(\hat{n})}\hat{a}^2$, where $F(\hat{n})$ is an operator-valued function of the number operator $\hat{n}$. We denote the eigenstates as $|\beta, F\rangle$ and they satisfy

$$\frac{1}{F(\hat{n})}\hat{a}^2\,|\beta, F\rangle = \beta\,|\beta, F\rangle\,. \tag{4}$$

In the Fock states $|n\rangle$ of the harmonic oscillator, $|\beta, F\rangle$ is given by

$$|\beta, F\rangle = \sum_{n=0}^{\infty} b_n \, |n\rangle. \tag{5}$$

Substituting (5) into (4), we have

$$b_{n+2} = \frac{\beta F(n) b_n}{\sqrt{(n+1)(n+2)}}, \tag{6}$$

where the function $F(n)$ is obtained by replacing the number operator $\hat{n}$ in $F(\hat{n})$ by the integer n. This recurrence relation among the expansion coefficients gives

$$b_{2n} = \frac{\beta^n F(2(n-1))!!}{\sqrt{(2n)!}} b_0, \quad b_{2n+1} = \frac{\beta^n F(2n-1)!!}{\sqrt{(2n+1)!}} b_1, \tag{7}$$

where

$$F(2(n-1))!! = F(0)F(2)F(4)\cdots F(2(n-1)), \quad F(0) = 1, \tag{8}$$

$$F(2n-1)!! = F(1)F(3)F(5)\cdots F(2n-1). \tag{9}$$

The constants b_0 and b_1 are fixed by normalization of the states $|\beta, F\rangle$. If we choose $b_1 = 0$, the state $|\beta, F\rangle$ involves the superposition of even number (Fock) states and represents the even NLCS. If $b_0 = 0$, the state $|\beta, F\rangle$, the superposition of odd number states, is the odd NLCS. We denote the even NLCS as $|\beta, F\rangle_+$ and the odd NLCS as $|\beta, F\rangle_-$. The even NLCS given by

$$|\beta, F\rangle_+ = b_0 \sum_{n=0}^{\infty} \frac{\beta^n F(2(n-1))!!}{\sqrt{(2n)!}} |2n\rangle, \quad |b_0|^{-2} = \sum_{n=0}^{\infty} \frac{|\beta|^{2n} [F(2(n-1))!!]^2}{(2n)!}. \tag{10}$$

Similarly, the odd NLCS given by

$$|\beta, F\rangle_- = b_1 \sum_{n=0}^{\infty} \frac{\beta^n F(2n-1)!!}{\sqrt{(2n+1)!}} |2n+1\rangle, \quad |b_1|^{-2} = \sum_{n=0}^{\infty} \frac{|\beta|^{2n} [F(2n-1)!!]^2}{(2n+1)!}. \tag{11}$$

Clearly, in the linear limit, $F(\hat{n}) = 1$, the even NLCS become the usual ECS and the odd NLCS become the usual OCS, respectively. Depending on the form of $F(\hat{n})$, this kind of the even and odd NLCSs may exhibit many of the nonclassical features.

If we take the operator function $F(\hat{n})$ to be $1\big/(1 + k\hat{n})$ (Sivakumar, 1998) (we choose this kind of nonlinearity function in the following numerical calculation), where $0 \le k \le 1$, the even NLCS is

$$|\beta, F\rangle_+ = b_{+1} \sum_{n=0}^{\infty} \frac{\beta^n}{(1 + k(2n-2))!!\sqrt{(2n)!}} |2n\rangle, \tag{12}$$

$$|b_{+1}|^{-2} = \sum_{n=0}^{\infty} \frac{|\beta|^{2n}}{[(1+k(2n-2))!!]^2(2n)!}, \tag{13}$$

and the odd NLCS is

$$|\beta, F\rangle_- = b_{-1} \sum_{n=0}^{\infty} \frac{\beta^n}{(1+k(2n-1))!!\sqrt{(2n+1)!}} |2n+1\rangle, \tag{14}$$

$$|b_{-1}|^{-2} = \sum_{n=0}^{\infty} \frac{|\beta|^{2n}}{[(1+k(2n-1))!!]^2(2n+1)!}. \tag{15}$$

3. QUADRATUE SQUEEZING OF EVEN AND ODD NLCSs

We consider the usual squeezing in terms of the quadrature operator X_1 and X_2 defined as

$$X_1 = (\hat{a}^+ + \hat{a})/2, \quad X_2 = i(\hat{a}^+ - \hat{a})/2, \tag{16}$$

such that $[X_1, X_2] = i/2$, which implies the uncertainty relation

$$\langle(\Delta X_1)^2\rangle\langle(\Delta X_2)^2\rangle \geq \frac{1}{16}. \tag{17}$$

Then the squeezing of the two operators may be conveniently described by the following two parameters

$$D(i) = 4\langle(\Delta X_i)^2\rangle - 1, \quad (i = 1, 2). \tag{18}$$

The quantities given in Equation(17) are also described in terms of the operators $\hat{a}$ and $\hat{a}^+$ as follows:

$$D(1) = 2\langle\hat{a}^+\hat{a}\rangle + \langle\hat{a}^{+^2} + \hat{a}^2\rangle - \langle\hat{a}^+ + \hat{a}\rangle^2, \tag{19}$$

$$D(2) = 2\langle\hat{a}^+\hat{a}\rangle - \langle\hat{a}^{+^2} + \hat{a}^2\rangle + \langle\hat{a}^+ + \hat{a}\rangle^2, \tag{20}$$

where $-1 \leq D(i) < 0$ for the usual squeezing of the field in the direction $X_i \quad (i = 1 \text{ or } 2)$, and the maximum squeezing (100%) is obtained when $D(i) = -1 \quad (i = 1$ or 2).

Now, we study the characteristics of the squeezing in the even and odd NLCSs [i.e., the states described by Equations(11) and (13)].

Using Equations(11) and (13), for the even and odd NLCSs, we have obtained the following expectation values of some operators:

$$\langle\hat{a}\rangle_{\pm} = 0, \quad \langle\hat{a}^+\rangle_{\pm} = 0, \tag{21}$$

$$\langle\hat{a}^+\hat{a}\rangle_+ = |b_{+1}|^2 \sum_{n=0}^{\infty} \frac{|\beta|^{2n}\, 2n}{[(1+k(2n-2))!!]^2(2n)!}, \tag{22}$$

$$\langle \hat{a}^{+}\hat{a}\rangle_{-} = |b_{-1}|^2 \sum_{n=0}^{\infty} \frac{|\beta|^{2n}(2n+1)}{[(1+k(2n-1))!!]^2(2n+1)!}, \tag{23}$$

$$\langle \hat{a}^{2}\rangle_{+} = \beta\, |b_{+1}|^2 \sum_{n=0}^{\infty} \frac{|\beta|^{2n}}{[1+k(2n-2)]!![1+2kn]!!(2n)!}, \tag{24}$$

$$\langle \hat{a}^{2}\rangle_{-} = \beta\, |b_{-1}|^2 \sum_{n=0}^{\infty} \frac{|\beta|^{2n}}{[1+k(2n-1)]!![1+k(2n+1)]!!(2n+1)!}. \tag{25}$$

The expectation value of $\langle \hat{a}^{+2}\rangle_{\pm}$ are the complex conjugate of $\langle \hat{a}^{2}\rangle_{\pm}$.

Substituting Equations(19)–(23) into (18), with the aid of a numerical method, the variations of functions $D(1)$ and $D(2)$ versus $|\beta|$ (when $\arg\beta = 0$ and $k = 0.75$) are shown in detail in Fig. 1.

From Fig. 1, it is evident that, in some ranges of $|\beta|$, the quadratue squeezing only consist in the even NLCS (in the direction X_2). For a fixed value ($k = 0.75$) of the parameter k, the ranges are $2.3105 < |\beta|$. The result shows that the quadratue squeezing properties of the new even NLCS are different from those of the usual ECS (Xia and Guo, 1989).

4. ANTIBUNCHING EFFECT OF EVEN AND ODD NLCSs

Now we study the antibunching effect of the even and odd NLCSs given by Equations(11)—(14). If the second-order correlation function of a light field

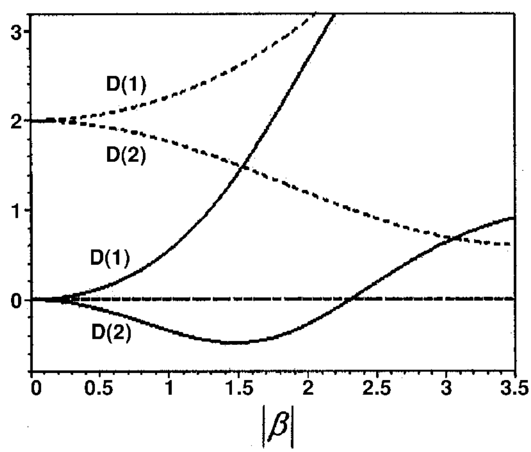

Fig. 1. Variation of the functions $D(1)$ and $D(2)$ with $|\beta|$ for $k = 0.75$. Solid curves (the even NLCS) and broken curves (the odd NLCS).

(Walls, 1983) is less than 1, i.e., $g^{(2)}(0) < 1$, one says that the light field exhibits an antibunching effect. In a similar way, we introduce the second-order correlation for the even and odd NLCSs,

$$g_{\pm}^{(2)}(0) = \frac{_{\pm}\langle\beta, F|\, \hat{a}^{+2}\hat{a}^{2}\, |\beta, F\rangle_{\pm}}{_{\pm}\langle\beta, F|\, \hat{a}^{+}\hat{a}\, |\beta, F\rangle_{\pm}^{2}}. \tag{26}$$

If $g_{+}^{(2)}(0) < 1$ (or $g_{-}^{(2)}(0) < 1$), we say that the new even (or odd) NLCS given by Equation(11) [or Equation(13)] exhibits the antibunching effect.

From Equations(11) and (13), we obtain

$$\langle\hat{a}^{+2}\hat{a}^{2}\rangle_{+} = |b_{+1}|^{2} \sum_{n=0}^{\infty} \frac{|\beta|^{2n}\,(2n)(2n-1)}{[(1+k(2n-2))!!]^{2}(2n)!}, \tag{27}$$

$$\langle\hat{a}^{+2}\hat{a}^{2}\rangle_{-} = |b_{-1}|^{2} \sum_{n=0}^{\infty} \frac{|\beta|^{2n}\,(2n+1)(2n)}{[(1+k(2n-1))!!]^{2}(2n+1)!}. \tag{28}$$

Substituting Equations(20), (21), (25) and (26) into (24), with the aid of a numerical method, the variations of functions $g_{\pm}^{(2)}(0)$ versus $|\beta|$ (when $\arg\beta = 0$ and $k = 0.75$) are shown in detail in Fig. 2.

From Fig. 2, we can see that, the odd NLCS always exhibits the antibunching effect for the arbitrary value of $|\beta|$, while the even NLCS may exhibits the antibunching effect in some wide ranges of $|\beta|$ for a fixed value of k. For example, for $k = 0.75$, except the ranges $|\beta| < 2.5625$, the even NLCS exhibits the antibunching effect. This means that the sub-Poissonian distributions (i.e., the antibunching

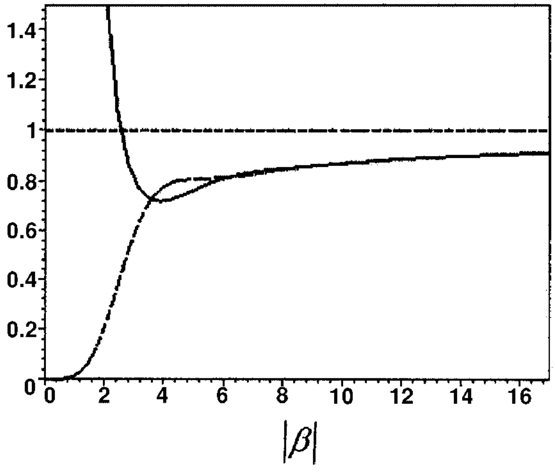

Fig. 2. Variation of the function $g_{\pm}^{(2)}(0)$ with $|\beta|$ for $k = 0.75$. Solid curve (the even NLCS) and broken curve (the odd NLCS).

effect) of the even NLCS are very different from those of the usual ECS (Xia and Guo, 1989).

5. PHASE PROBABILITY DISTRIBUTION OF EVEN AND ODD NLCSs

It is well known that the phase probability distribution is an essential tool in the study of various phase characteristics. In this section, we now turn to the phase probability distributions for the even and odd NLCSs given by Equations(11) and (13). According to the Pegg–Barnett phase operator formalism (Barnett and Pegg, 1989; Pegg and Barnett, 1988, 1989) we start with a finite dimensional $(s+1)$ Hilbert space spanned by the number states $|0\rangle, |1\rangle, \cdots, |s\rangle$. In this space a complete orthonormal set of phase states $|\theta_m\rangle$, $m = 0, 1, 2, \cdots, s$, is defined by

$$|\theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{in\theta_m} |n\rangle, \tag{29}$$

where θ_m are given by

$$\theta_m = \theta_0 + \frac{2m\pi}{s+1}, \quad m = 0, 1, 2, \cdots, s. \tag{30}$$

The value of θ_0 is arbitrary and defines a particular basis in the phase space. In this space a hermitian phase operator Φ_θ is defined as

$$\Phi_\theta = \sum_{m=0}^{s} \theta_m |\theta_m\rangle \langle\theta_m|. \tag{31}$$

For superposition states of the form $|\psi\rangle = \sum_{n=0}^{\infty} C_n e^{in\varphi} |n\rangle$ the phase probability distribution is given by

$$|\langle\theta_m \mid \psi\rangle|^2 = \frac{1}{s+1} + \frac{2}{s+1} \sum_{n>n'} C_n C_{n'} \cos[(n-n')(\varphi - \theta_m)]. \tag{32}$$

Choosing θ_0 as $\theta_0 = \varphi - \frac{s\pi}{s+1}$, we obtain from Equation(29)

$$|\langle\theta_m \mid \psi\rangle|^2 = \frac{1}{s+1} + \frac{2}{s+1} \sum_{n>n'} C_n C_{n'} \cos\left[(n-n')\frac{2\mu\pi}{s+1}\right], \tag{33}$$

where $\mu = m - s/2$. The continuous phase probability distribution $P(\theta)$ can now be obtained as

$$\begin{aligned} P(\theta) &= \lim_{s\to\infty} \frac{s+1}{2\pi} |\langle\theta_m \mid \psi\rangle|^2 \\ &= \frac{1}{2\pi}\left(1 + 2\sum_{n>n'} C_n C_{n'} \cos[(n-n')\theta]\right), (-\pi \le \theta \le \pi). \end{aligned} \tag{34}$$

For the even and odd NLCSs given by Equations(11) and (13), the continuous phase probability distribution $P_{\pm}(\theta)$ is given by

$$P_{\pm}(\theta) = \frac{1}{2\pi}\left(1 + 2\sum_{n>n'}(C_{\pm})_n(C_{\pm})_{n'}\cos[(n-n')\theta]\right), (-\pi \leq \theta \leq \pi), \quad (35)$$

where

$$(C_{\pm})_n = N_{\pm}N_f \frac{[r^n \pm (-r)^n]F(n)!}{\sqrt{n!}},$$

$$N_{\pm} = \left\{2 \pm 2N_f^2 \sum_{n=0}^{\infty} \frac{(-|\beta|^2)^n[F(n)!]^2}{n!}\right\}^{-1/2}, \quad (36)$$

where $\beta = r\exp(i\varphi)$. The results of numerical computations of the continuous phase probability distribution for the even and odd NLCSs are presented in Figs. 3 and 4.

In Figs. 3 and 4, for the even and odd NLCSs, we plot the phase probability distribution keeping k fixed at 0.75 and varying $|\beta|$ ($|\beta| = 0.4$, 0.6 and 0.8). From the figures we find that the phase probability distribution of the even and odd NLCSs remain essentially the same for different values of $|\beta|$ when k is kept fixed. The distributions all have only a central peak at $\theta = 0$. However, it may be noted that as $|\beta|$ increases the peak structure becomes more and more prominent. On the other hand, from the figures it is seen that, for the fixed value of $|\beta|$, the peak structure of the phase probability distribution of the odd NLCS has become more prominent than those of the even NLCS. Therefore, unlike the case of ordinary even

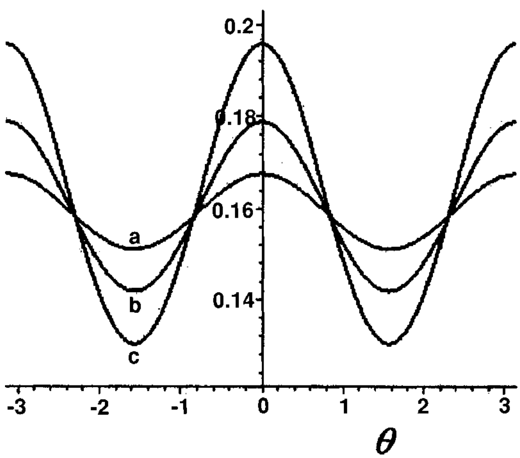

Fig. 3. Phase distribution of the even NLCS for $k = 0.75$ and $|\beta| = 0.4$ (curve a), 0.6 (curve b) and 0.8 (curve c).

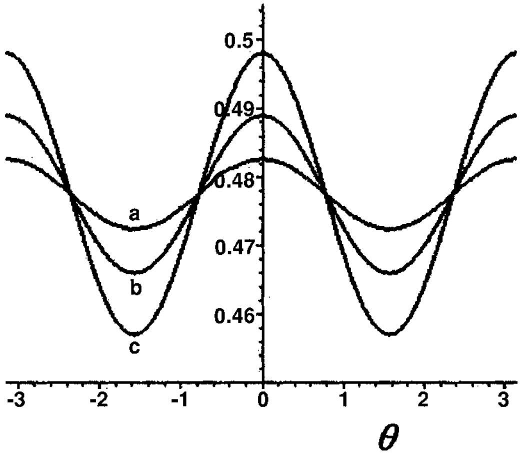

Fig. 4. Phase distribution of the odd NLCS for $k = 0.75$ and $|\beta| = 0.4$ (curve a), 0.6 (curve b) and 0.8 (curve c).

and odd CSs the Pegg–Barnett distribution clearly reflects the different character of quantum interference in the case of the new even and odd NLCSs.

6. CONCLUSIONS

In this paper, we introduced a class of even and odd NLCSs. The sum of their properties, including quadrature squeezing, antibunching effect and phase probability distribution, are investigated. This class of even and odd NLCSs has rather different statistical properties from those of the usual even and odd CSs. It is found that, for a fixed value ($k = 0.75$) of the parameter k, the squeezing only consists in the even NLCS, and the antibunching effect appear for both even and odd NLCSs in some ranges of $|\beta|$. The phase probability distribution of the even and odd NLCSs remain essentially the same for different values of $|\beta|$ when k is kept fixed. However, for the fixed value of $|\beta|$, the peak structure of the phase probability distribution of the odd NLCS has become more prominent than those of the even NLCS. Therefore, unlike the case of ordinary even and odd CSs the Pegg–Barnett distribution clearly reflects the different character of quantum interference in the case of the even and odd NLCSs.

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